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DISCRETE HYPERBOLIC GEOMETRY

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The aim of the paper is to make geometers and combinatorialists familiar with old and new connections between the geometry of Lorentz space and combinatorics. Among the topics treated are equiangular lines and their relations to spherical 2-distance sets; spherical and hyperbolic root systems and their relation to graphs whose second largest eigenvalue does not exceed one or two, respectively; and work of Niemeier, Vinberg, Conway and Sloane on Euclidean and Lorentzian unimodular lattices.

1. Introduction

There exist close relations between combinatorics and real geometry. An example of such a relation is provided by the following. For any graph on n vertices with adjacency matrix A and largest eigenvalue λ_1 we may regard $\lambda_1 I - A$ as the Gram matrix of the positive definite inner products of n vectors of equal norm λ_1. This situation may be generalized in two ways, to arbitrary $\lambda \in \mathbf{R}$ and to arbitrary symmetric matrices M. Then $\lambda I - M$ is the Gram matrix of the possibly indefinite inner products of n vectors. Since many interesting matrices M have λ_1 only of multiplicity one (for instance the Perron eigenvalue of an irreducible nonnegative matrix), it seems worthwhile to concentrate on the second largest eigenvalue λ_2 and to consider $\lambda_2 I - M$ as the Gram matrix of vectors in the indefinite space $\mathbf{R}^{p,1}$. This is particularly useful if λ_2 has a large multiplicity, as for example may occur for strongly regular graphs, with spectrum $\lambda_1^1, \lambda_2^{n-1-p}, \lambda_3^p$.

More generally, the aim of the present paper is to investigate the applicability of the geometry of Lorentz space $\mathbf{R}^{p,1}$ to combinatorics. Why restrict the indefinite space to $\mathbf{R}^{p,1}$? Indeed, $\mathbf{R}^{p,1}$ seems to be preeminently suitable for combinatorial applications. Apart from the material presented in the present paper, this statement is supported by the following results in the literature (for the proofs we refer to the respective references).

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[32]: $\mathbf{R}^{p,0}$ and $\mathbf{R}^{1,q}$ are characterized as the positive inner product spaces over $\mathbf{R}$ in which every positive subspace is nondegenerate (a vector space is positive if it contains at least one vector with positive norm).

[22]: For vectors $\mathbf{a}_1, \ldots, \mathbf{a}_{p-1}$ with positive coordinates in $(p+1)$-space, the permanent $-$ per $(\mathbf{a}_1, \ldots, \mathbf{a}_{p-1}, \mathbf{x}, \mathbf{y})$ defines a Lorentz inner product. This leads to the key inequality for the proof of van der Waerden's conjecture for permanents.

[20]: The incidence matrix of a symmetric design (v, k, λ) may be extended to an orthogonal matrix in $\mathbf{R}^{v,1}$, which is related to selforthogonal codes over $GF(p)$, where p divides $k-\lambda$.

[30]: Nikulin shows that there are only finitely many even integral reflexive lattices of signature $(p, 1)$, $p \geqq 4$. In contrast to this, Kneser [18] shows that a very large class of integral lattices of signature (p, q), $p \geqq 2$, $q \geqq 2$ is reflexive.

We now turn to a short description of the sections of this paper. Lorentz space $\mathbf{R}^{p,1}$ is introduced in Section 2. We are mainly interested in the geometry outside of the light cone. This provides, for instance, descriptions of the root system E_8 and of convex polyhedra in Bolyai—Lobachewsky space, in connection with discrete groups generated by reflections [11, 39]. Section 3 describes the lifting of Euclidean $\mathbf{R}^p$ onto a p-flat in $\mathbf{R}^{p+1}$ or in $\mathbf{R}^{p,1}$. As an application, spherical 2-distance sets corresponding to interesting graphs are lifted into sets of equiangular lines in Euclidean [23] or in Lorentz space. Graphs with $\lambda_2 = 1$ are treated in Section 4, which includes old results by Du Val [15] and Coxeter [12]. The complements of the root system graphs [6] having smallest eigenvalue — 2 belong to this class, but there are others. In Section 5 come the reflexive graphs, with $\lambda_2 = 2$. We give an account of the examples by Vinberg and others [40, 25] of sets of unit vectors at 90° and 120° in $\mathbf{R}^{p,1}$, $p \leqq 19$. This motivates the study of a particular extension problem for graphs (for sets of vectors in $\mathbf{R}^{p,1}$). Three important integral unimodular Euclidean lattices, in the dimensions 8, 19, 24, are considered in Section 6, also in connection with recent work by Conway and Sloane [7, 8, 9]. The vectors of norm 2, 3, 4, respectively, in these lattices may provide a source for the construction of interesting graphs. Unimodular integral lattices H of signature $(p, 1)$, the subject of Section 7, are isomorphic to $\mathbf{I}_{p,1}$ in the odd, and to $\mathbf{II}_{p,1}$ in the even case. Their interest stems from the construction by use of "stereographic projection" of the Euclidean lattices $\mathbf{u}^\perp/\langle\mathbf{u}\rangle$, depending on the istoropic indivisible $\mathbf{u} \in H$. We survey work initiated by Vinberg [38], and mention recent results by Conway—Sloane [10] and by Neumaier [28].

The style of the present manuscript is informal. No theorems occur in the text; too formal discussions have been avoided. Yet the paper is not merely a collection of facts. We do indicate mathematical reasoning, and at various occasions we suggest shortcuts in the proofs. Our main aim is to introduce the subject and its methods, to survey the existing literature, to pose problems, to indicate connections, and to suggest possibilities for applications.

2. Lorentz space

Any real symmetric matrix G of size n and rank d may be viewed as the Gram matrix of the inner products of n vectors in $\mathbf{R}^d$. Indeed, G may be diagonalized by an orthogonal matrix M; the diagonals Λ^+ of the p positive and Λ^- of the q negative eigenvalues can be made into the unit matrices I_p and $-I_q$:

$$G = M\begin{bmatrix} \Lambda^+ & 0 & 0 \\ 0 & \Lambda^- & 0 \\ 0 & 0 & 0 \end{bmatrix} M^t = N\begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} N^t.$$

The rows of the $n\times d$ matrix N are interpreted as n vectors in $\mathbf{R}^d$, for which G is the Gram matrix of the inner products

$$(\mathbf{x}, \mathbf{y}) := x_1 y_1 + \ldots + x_p y_p - x_{p+1} y_{p+1} - \ldots - x_d y_d.$$

We denote this vector space by $\mathbf{R}^{p,q}$, with $p+q=d$. The above is esentially Sylvester's law of inertia.

Lorentz space is defined to be the indefinite inner product space $\mathbf{R}^{p,1}$, that is, $\mathbf{R}^{p,q}$ with $q=1$. We use coordinates $\mathbf{x}=(x_0; x_1, \ldots, x_p)$, and

$$(\mathbf{x}, \mathbf{y}) = -x_0 y_0 + x_1 y_1 + \ldots + x_p y_p.$$

Lines $\langle \mathbf{a} \rangle$ through the origin are of three types: positive (spacelike), negative (timelike) and isotropic, according to the sign of $(\mathbf{a}, \mathbf{a})$. We say that they are outside, inside, and on the light cone

$$\{x \in \mathbf{R}^{p,1} | x_0^2 = x_1^2 + \ldots + x_p^2\},$$

respectively. Also planes $\langle \mathbf{a}, \mathbf{b} \rangle$ are of three types: intersecting, passing, and tangent with respect to the light cone, according as the equation

$$(\alpha\mathbf{a}+\beta\mathbf{b},\ \alpha\mathbf{a}+\beta\mathbf{b}) = \alpha^2(\mathbf{a}, \mathbf{a}) + 2\alpha\beta(\mathbf{a}, \mathbf{b}) + \beta^2(\mathbf{b}, \mathbf{b}) = 0$$

has 2, 0, 1 solutions, that is, according as $(\mathbf{a}, \mathbf{b})^2 - (\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b})$ is positive, negative, zero, respectively. For half lines inside one nappe of the light cone this leads to Bolyai—Lobachewsky space Λ^p of dimension p, in which the points $A=\langle \mathbf{a} \rangle$, $B=\langle \mathbf{b} \rangle$ have the distance d_{AB} defined by

$$\cosh d_{AB} = \frac{(\mathbf{a}, \mathbf{b})}{\sqrt{(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b})}}.$$

On the other hand, for lines $A=\langle \mathbf{a} \rangle$ and $B=\langle \mathbf{b} \rangle$ in a passing plane the angle φ_{AB} is defined by

$$\cos \varphi_{AB} = \frac{|(\mathbf{a}, \mathbf{b})|}{\sqrt{(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b})}}.$$

As in real and complex Euclidean spaces [14], we may investigate sets of points with few distances in Bolyai—Lobachewsky space Λ^p, and sets of lines with few angles outside the light cone of Lorentz space $\mathbf{R}^{p,1}$. Bannai and others [2] obtained

$$|X| \leqq \binom{p+s}{s}$$

for the cardinality of an s-distance subset X of Λ^p. Blokhuis [4] obtained the bounds $|Y| \leq \binom{p+1}{2}$ for a set Y of equiangular lines in $\mathbf{R}^{p,1}$, and $|Z| \leq \binom{p+2}{3}$ for a set Z of lines $\mathbf{R}^{p,1}$, each pair of which has an angle φ with $\cos\varphi \in \{0, \alpha\}$. Easy examples meeting these bounds are provided by $Y = \{(1/2\sqrt{2};\ 1^2, 0^{p-1})\}$, for $p > 7$, which are all orthogonal to $(2\sqrt{2};\ 1^{p+1})$, and $Z = \{(1;\ 1^3, 0^{p-2}), (0;\ 1, -1, 0^{p-1})\}$, for $p > 9$, which all are orthogonal to $(3;\ 1^{p+1})$. In both cases, the restriction for p ensures that the orthogonal subspace remains Lorentzian.

Also hyperplanes in $\mathbf{R}^{p,1}$ are of three types: Lorentzian, Euclidean, and degenerate Euclidean hyperplanes. Indeed, each hyperplane may be represented as

$$\mathbf{a}^{\perp} := \{\mathbf{x} \in \mathbf{R}^{p,1} | (\mathbf{x}, \mathbf{a}) = 0\},$$

with the vector $\mathbf{a}$ outside, inside, and on the light cone, respectively. In the last case a Euclidean space of dimension $p-1$ is provided by the quotient space $\mathbf{a}^{\perp}/\langle \mathbf{a} \rangle$. For example, in $\mathbf{R}^{9,1}$ the $120 = 84 + 36$ lines spanned by $(1;\ 1^3, 0^6)$ and $(0;\ 1, -1, 0^7)$ are orthogonal to the isotropic vector $(3;\ 1^9)$, have angles 90° and 60°, and hence constitute the root system E_8 in Euclidean $\mathbf{R}^8$, cf. [6].

Let $\mathbf{a}$ and $\mathbf{b}$ be independent unit vectors: $\mathbf{a} \pm \mathbf{b} \neq \mathbf{0}$, $(\mathbf{a}, \mathbf{a}) = (\mathbf{b}, \mathbf{b}) = 1$. Then $\mathbf{a}^{\perp} \cap \mathbf{b}^{\perp}$ is intersecting, passing, or tangent with respect to the light cone according as $\langle \mathbf{a}, \mathbf{b} \rangle$ is a passing, intersecting, or tangent plane, respectively, that is, according as $(\mathbf{a}, \mathbf{b})^2$ is < 1, > 1, $= 1$. In the case of a passing plane $\langle \mathbf{a}, \mathbf{b} \rangle$ the dihedral angle φ between the hyperplanes $\mathbf{a}^{\perp}$ and $\mathbf{b}^{\perp}$ is the supplement of the angle between $\mathbf{a}$ and $\mathbf{b}$ that is, $\cos(\pi - \varphi) = (\mathbf{a}, \mathbf{b})$.

For a finite set of unit vectors $\mathbf{a}_i$, $i \in I$, $(\mathbf{a}_i, \mathbf{a}_i) = 1$, we consider the intersection of the halfspaces $H_i := \{x \in \mathbf{R}^{p,1} | (\mathbf{x}, \mathbf{a}_i) \leq 0\}$. If $\bigcap_{i \in I} H_i$ is properly contained in the light cone, then the corresponding subset of Bolyai—Lobachewsky space Λ^p is a bounded convex polyhedron. If $\bigcap_{i \in I} H_i$ is contained in the light cone joined with its closure, then the corresponding subset of Λ^p is still a convex polyhedron (although it may have infinite points). Thus convex polyhedra in Λ^p are described by finite sets of unit vectors in $\mathbf{R}^{p,1}$.

Andreev [1] proved the following result. If for a convex polyhedron in Λ^p all pairs of adjacent faces have dihedral angle $\leq \pi/2$, then the pairs of nonadjacent faces do not intersect in Λ^p (i.e. are parallel or diverging). Hence if the pairs of unit vectors corresponding to adjacent faces have $(\mathbf{a}_i, \mathbf{a}_j) \leq 0$, then the pairs corresponding to nonadjacent faces have $(\mathbf{a}_\mu, \mathbf{a}_\nu) \leq -1$ (since they must be opposite). In this situation the Gram matrix of G of the unit vectors has signature $(p, 1)$ and determines which pairs of faces are adjacent and nonadjacent, respectively. This applies for example to discrete groups generated by reflections, as we shall see in Section 7.

3. Lifting

In this section we lift Euclidean $\mathbf{R}^p$ into the space $\mathbf{R}\mathbf{w} \oplus \mathbf{R}^p$ of dimension $p+1$, by use of a vector $\mathbf{w} \perp \mathbf{R}^p$ of norm $(\mathbf{w}, \mathbf{w}) = d \neq 0$. Hence $\mathbf{R}\mathbf{w} \oplus \mathbf{R}^p$ is isomorphic to $\mathbf{R}^{p+1}$ if $d > 0$, and to $\mathbf{R}^{p,1}$ if $d < 0$. The map $\mathbf{x} \to \mathbf{y} = \mathbf{x} + \mathbf{w}$ lifts $\mathbf{R}^p$ onto the p-dimensional flat

$$\{\mathbf{y} \in \mathbf{R}\mathbf{w} \oplus \mathbf{R}^p | (\mathbf{y}, \mathbf{w}) = d\}.$$

Since $(\mathbf{y}_1, \mathbf{y}_2)=(\mathbf{x}_1, \mathbf{x}_2)+d$, we have the possibility of lifting a set X of unit vectors with angles ξ from a given set onto a set Y of vectors with different angles η given by

$$\cos\eta = \frac{d+\cos\xi}{d+1}.$$

To do so, it is necessary that $d \geqq -(1+\cos\xi)/2$ for all occurring ξ.

We apply the method to a spherical two-distance set X with two angles ξ_1 and ξ_2, cf. [23]. We wish to lift X onto a set Y of vectors which span equiangular lines, that is, which have $\eta_1+\eta_2=\pi$. In the nontrivial case this is achieved by taking $2d = -\cos\xi_1-\cos\xi_2$. Thus, any spherical two-distance set in $\mathbf{R}^p$ is lifted into a set of equiangular lines in $\mathbf{R}^{p+1}$ or in $\mathbf{R}^{p,1}$, according as $\cos\xi_1+\cos\xi_2$ is negative or positive. A well-known example is provided by lifting the vertices of the regular pentagon in $\mathbf{R}^2$ into the diagonals of a regular icosahedron in $\mathbf{R}^3$ (the sixth diagonal orthogonal to $\mathbf{R}^2$ is obtained for free).

Many further examples may be obtained from graphs. The lifting process then amounts to the following. Let A be the (0, 1)-adjacency matrix of any regular graph of order n and valency k. Let $k=\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$ be its eigenvalues, and let $n-1-p$ be the multiplicity of λ_n. Then

$$A-\lambda_n I-\frac{k-\lambda_n}{n}J$$

is positive semi-definite of rank p, and may be viewed as the Gram matrix of a spherical two-distance set in $\mathbf{R}^p$. On the other hand, the graph has $C:=2A-J+I$ as its $(1, -1)$ adjancency matrix, with the eigenvalues $\gamma_1=2k+1-n$, $\gamma_2=2\lambda_2+1, \ldots, \gamma_n=2\lambda_n+1$. Now $C-\gamma_n I$ is the Gram matrix of n vectors, which span a set of n equiangular lines at $\cos\eta=-1/\gamma_n$ in $\mathbf{R}^{p+1}$ or in $\mathbf{R}^{p,1}$, depending on whether $\gamma_1-\gamma_n = 2k-2\lambda_n-n$ is positive or negative (in the case $\gamma_1=\gamma_n$ the two-distance set itself yields equiangular lines in $\mathbf{R}^p$). Special examples are obtained from strongly regular graphs [34], that is, when A has only three eigenvalues k, r, s of multiplicities $1, p, n-1-p$, say. The following examples are obtained from well-known rank 3 graphs [34], and yield sets of n equiangular lines at $\cos\eta$ in $\mathbf{R}^{p+1}$ or in $\mathbf{R}^{p,1}$. Note that in case of $T(8)$ the line system Y has the bigger automorphism group $Sp(6, 2)$.

group of X	S_5	2^4S_5	$O_6^-(2)$	S_8	S_{10}	Sz	Fi_{23}
n	10	16	27	28	45	416	31671
k	6	10	16	12	16	100	3510
s	-2	-2	-2	-2	-2	-4	-9
$(\cos\eta)^{-1}$	3	3	3	3	3	7	17
dim	$4+1$	$5+1$	$6+1$	7,0	9,1	65,1	782,1
name	Petersen	Clebsch	Schläfli	$T(8)$	$T(10)$	Suzuki	Fischer

The preceding examples form a particular case of the following situation. Let $X\subset\mathbf{R}^p$ be an n-set whose center of mass is in the origin, hence whose Gram matrix has zero row sums. We apply the transformation $\mathbf{y}=\mathbf{x}+\mathbf{w}$, with $(\mathbf{w}, \mathbf{w})=d$, in order to lift X onto an n-set Y in $\mathbf{R}^{p+1}$ or in $\mathbf{R}^{p,1}$. Then $n\mathbf{w}=\sum_{\mathbf{y}\in Y}\mathbf{y}$ and, for any $\mathbf{y}_0\in Y$,

$$\sum_{\mathbf{y}\in Y}(\mathbf{y},\mathbf{y}_0)=(n\mathbf{w},\mathbf{y}_0)=n(\mathbf{w},\mathbf{x}_0+\mathbf{w})=nd.$$

Hence the Gram matrix of Y has constant row sums, positive or negative depending on whether the space is definite or not. It is easily seen that, conversely, any set Y in $\mathbf{R}^{p+1}$ or in $\mathbf{R}^{p,1}$ having constant row sums of the appropriate sign can be lifted from a set in $\mathbf{R}^p$. This leads to the question of whether it is possible to span $\mathbf{R}^{p,1}$ by a set Y having constant positive row sums.

In the case mentioned in the introduction, the Gram matrix of Y has the form $\lambda I-A$, where A is the adjacency matrix of a graph Γ. The row sums are constant $\lambda-k$ iff Γ is regular of valency k. In particular if $\lambda_2\leqq\lambda<k$ then the row sum is negative and Y is in the hyperplane $(\mathbf{w},\mathbf{y}-\mathbf{w})=0$ of $\mathbf{R}^{p,1}$; in fact, Y is lifted from a set X in $\mathbf{R}^p$ with Gram matrix $\lambda I-A+\frac{k-\lambda}{n}J$.

4. Graphs with $\lambda_2=1$

Let $r\in\mathbf{R}$, and let A be the (0, 1) adjacency matrix of a graph with the eigenvalues $\lambda_1\geqq\lambda_2\geqq\ldots\geqq\lambda_n$. Then $rI-A$ is the Gram matrix of n vectors in some $\mathbf{R}^{p,q}$ where $n-p-q$ is the multiplicity of r as an eigenvalue of A. We have n vectors in Euclidean space $\mathbf{R}^p$ if $r\geqq\lambda_1$, and n vectors in Lorentz space $\mathbf{R}^{p,1}$ if $\lambda_1>r\geqq\lambda_2$. This applies in particular for strongly regular graphs with spectrum k^1, r^{n-p-1}, s^p (notice that now p is the multiplicity of the smallest eigenvalue). All such graphs with $r=1$ are known, as well as several with $r=2$, cf. [17]. We first look at the case $r=1$.

The only graphs having positive semidefinite $I-A$ are the one-factors, represented in $\mathbf{R}^p$ by the cross-polytope

$$B^{(p)}:=\{\pm\mathbf{e}_1,\ldots,\pm\mathbf{e}_p\},$$

and their subgraphs. For graphs with $\lambda_2=1$ we look for sets of n unit vectors in $\mathbf{Z}^{p,1}$ having Gram matrix $I-A$, that is, having inner products -1 or 0 according as the corresponding vertices are adjacent or not. Such unit vectors can be found, for example, in the truncated cross-polytope

$$D^{(p)}:=\{\mathbf{e}_0\pm\mathbf{e}_i\pm\mathbf{e}_j\,|\,1\leqq i\neq j\leqq p\}.$$

Each of these $2p(p-1)$ vectors has norm 1, and their mutual inner products are in $\{0,-1,-2,-3\}$. Thus we need subsets in which -2 and -3 do not appear. Among the graphs represented by subsets of $D^{(p)}$ are the complements of the generalized line graphs [6]. In fact, the subset $\{\mathbf{e}_0-\mathbf{e}_i-\mathbf{e}_j\,|\,i,j\in V,\ \{i,j\}\in E\}$ provides the complement of the line graph of the graph (V,E). We may hang on one-factors of the type $\mathbf{e}_0-\mathbf{e}_i\pm\mathbf{e}_{i,a}$, $a=1,\ldots,a_i$, so as to obtain the complement of the generalized line graph $L((V,E);a_1,\ldots,a_p)$, cf. [6].

Apart from $D^{(p)}$, other interesting sets of unit vectors in $\mathbf{Z}^{p,1}$ are available, as was observed by Du Val [15] and Coxeter [12]. Let

$$H^{(p)}:=\{\mathbf{x}\in\mathbf{Z}^{p,1}\,|\,(\mathbf{x},\mathbf{x})=1=(\mathbf{x},\mathbf{w}^{(p)})\},$$

with $\mathbf{w}^{(p)} := -3\mathbf{e}_0+\mathbf{e}_1+\ldots+\mathbf{e}_p$. For $p=k+4$, this is isomorphic to Coxeter's polytope k_{21}, cf. [13] chapter 11. For $p \leqq 8$ the set $H^{(p)}$ is finite, and consists of all vectors of the following types (apply a permutation of $\{1, \ldots, p\}$ to the indices).

type of vector	$\mathbf{R}^{3,1}$	$\mathbf{R}^{4,1}$	$\mathbf{R}^{5,1}$	$\mathbf{R}^{6,1}$	$\mathbf{R}^{7,1}$	$\mathbf{R}^{8,1}$
$\mathbf{e}_1$	3	4	5	6	7	8
$\mathbf{e}_0-\mathbf{e}_1-\mathbf{e}_2$	3	6	10	15	21	28
$2\mathbf{e}_0-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3-\mathbf{e}_4-\mathbf{e}_5$			1	6	21	56
$3\mathbf{e}_0-2\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3-\mathbf{e}_4-\mathbf{e}_5-\mathbf{e}_6-\mathbf{e}_7$					7	56
$4\mathbf{e}_0-2\mathbf{e}_1-2\mathbf{e}_2-2\mathbf{e}_3-\mathbf{e}_4-\mathbf{e}_5-\mathbf{e}_6-\mathbf{e}_7-\mathbf{e}_8$						56
$5\mathbf{e}_0-2\mathbf{e}_1-2\mathbf{e}_2-2\mathbf{e}_3-2\mathbf{e}_4-2\mathbf{e}_5-2\mathbf{e}_6-\mathbf{e}_7-\mathbf{e}_8$						28
$6\mathbf{e}_0-3\mathbf{e}_1-2\mathbf{e}_2-2\mathbf{e}_3-2\mathbf{e}_4-2\mathbf{e}_5-2\mathbf{e}_6-2\mathbf{e}_7-2\mathbf{e}_8$						8
total numbers of vectors	6	10	16	27	56	240
Coxeter [13] notation	prism	$t\alpha_4$	$h\gamma_5$	2_{21}	3_{21}	4_{21}

For $p \geqq 9$ the set $H^{(p)}$ is infinite. Indeed, then $H^{(p)}$ contains e.g. the vectors $-4a^2\mathbf{w}^{(9)} + a(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4-\mathbf{e}_5-\mathbf{e}_6-\mathbf{e}_7-\mathbf{e}_8)+\mathbf{e}_9$ for all integers a. However, for $p \leqq 8$ the set $H^{(p)}$ is the finite set described above. Indeed, $\mathbf{x}=(x_0; x_1, \ldots, x_p) \in H^{(p)}$ iff $-x_0^2+x_1^2+\ldots+x_p^2=1$, $3x_0+x_1+\ldots+x_p=1$, and this implies

$$0 \leqq (3x_1+x_0)^2+\ldots+(3x_p+x_0)^2 = 9+6x_0+(p-9)x_0^2 \leqq 18-(x_0-3)^2,$$

leaving a finite amount of checking.

The inner products $\neq 1$ arising in $H^{(p)}$ are $\{0, -1\}$ for $p \leqq 6$, $\{0, -1, -2\}$ for $p=7$, and $\{0, -1, -2, -3\}$ for $p=8$. The set $H^{(3)}$ corresponds to the hexagon, and $H^{(4)}$, $H^{(5)}$, $H^{(6)}$ to the strongly regular graphs of Petersen, Clebsch, Schläfli, respectively, which were mentioned in Section 3. With $H^{(7)}$ we obtain the double cover of K_{28} corresponding to the regular two-graph on 28 vertices, cf. [33]. Indeed, the inner product -2 connects the $7+21$ antipodal pairs $\mathbf{e}_i$ and $-\mathbf{w}^{(7)}-\mathbf{e}_i$; $\mathbf{e}_0-\mathbf{e}_i-\mathbf{e}_j$ and $-\mathbf{w}^{(7)}-\mathbf{e}_0+\mathbf{e}_i+\mathbf{e}_j$. The subsets avoiding -2 yield graphs, in particular those of Chang and Shrikhande [6]. The 240 vectors of $H^{(8)}$ correspond to the 240 vectors of norm 2 in the exceptional root system E_8. Indeed, $\mathbf{w}=\mathbf{w}^{(8)}$ satisfies $(\mathbf{w}, \mathbf{w})=-1$, hence $\{\mathbf{w}+\mathbf{x} \mid \mathbf{x} \in H^{(8)}\} \subset \mathbf{w}^{\perp}$ is in Euclidean 8-space, and consists of 240 vectors of norm 2 having inner products $\in \{0, \pm 1, \pm 2\}$. This determines E_8 uniquely [3].

Thus we have found three classes of graphs with $\lambda_2 \leqq 1$, represented in $\mathbf{Z}^{p,1}$ by subsets of $B^{(p)}$, $D^{(p)}$ and $H^{(8)}$. It is well-known that the complements of these graphs have smallest eigenvalue $\geqq -2$; in fact, they are characterized by this property, cf. [6]. Any regular graph A with $\lambda_2 \leqq 1$ has this property. Indeed, the complement $\bar{A}$ has eigenvector j, and all eigenvalues are $\geqq -2$ since $I-A=I-(J-I-\bar{A})= =2I+\bar{A}-J$. However, there do exist non-regular graphs with $\lambda_2 \leqq 1$ whose complement has smallest eigenvalue < -2, for instance the graph consisting of a 5-clique and an isolated vertex. More generally,

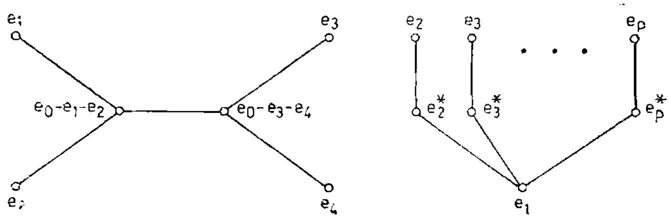
$$C^{(p)} := \{\mathbf{e}_i, \mathbf{e}_0-\mathbf{e}_i-\mathbf{e}_j \mid i, j = 1, \ldots, p\}$$

consists of unit vectors having mutual inner products $\in \{0, -1\}$. For $p \geqq 2m+1$, let Γ be a graph represented by a subset of $C^{(p)}$ which contains the $(m+1)$-set

$$\{\mathbf{e}_{2m+1}, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_{m+i} | i = 1, \ldots, m\}.$$

The complement $\bar{\Gamma}$ contains an m-claw, hence has smallest eigenvalue $\leqq -\sqrt{m}$.

We close this section with some remarks about trees. It was shown in [27] that trees having $\lambda_2 \leqq 1$ are of two types (which are both in $C^{(p)}$): either $\tilde{D}_5$, or $G(\mathbf{e}_1; \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_p)$ and its connected subtrees:



$\tilde{D}_5$

$\mathbf{e}_i^* = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_i$
$G(\mathbf{e}_1; \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_p)$

In view of the next Section 5 the following observation is of interest. The Petersen graph contains the subtree $\tilde{D}_5$, and also the subtree $G(\mathbf{e}_1; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$. The Clebsch graph contains $G(\mathbf{e}_1; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5)$, and the Schläfli graph contains $G(\mathbf{e}_1; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6)$. In fact, these graphs are extensions of the mentioned trees, in the sense of Section 5 below.

5. Reflexive graphs

After having discussed graphs A for which $I-A$ is the Gram matrix of vectors in $\mathbf{R}^p$ or in $\mathbf{R}^{p,1}$, we turn to $2I-A$. It is not difficult to determine all graphs having largest eigenvalue $\lambda_1 \leqq 2$. These are the extended Dynkin graphs of type A_n, D_n, E_n, their disjoint unions and their subgraphs. Graphs having $\lambda_2 \leqq 2$ correspond to sets of vectors in $\mathbf{R}^{p,1}$ having Gram matrix $2I-A$, that is, vectors of norm 2 and at angles 90° or 120°. We call such graphs reflexive graphs; they are Lorentzian counterparts of the spherical and the Euclidean graphs which occur in the theory of reflection groups.

More generally, a (symmetric) hyperbolic Cartan matrix [5] is a matrix $2I-M$ with symmetric, integral, nonnegative M, having one negative eigenvalue. Koszul [19] enumerated the minimal Cartan matrices of order n, that is, those which are nonsingular, irreducible, and having all principal minors of order $n-1$ positive semidefinite. Such a matrix is called compact if all these minors are positive definite. It turns out [19, 39] that there are no minimal hyperbolic Cartan matrices of size >10, and no compact ones of size >6.

Dropping the condition of minimality, we consider some further examples of reflexive graphs. They play an important rôle in the work by Vinberg [37, 38, 39, 40] on discrete groups generated by reflections in $\mathbf{R}^{p,1}$, as we shall see in Section 7.

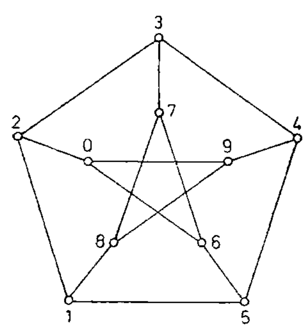
Let N be the 10×15 vertex-edge incidence matrix of the Petersen graph and let P be its 10×10 adjacency matrix. Then

$$NN^t = 3I+P$$

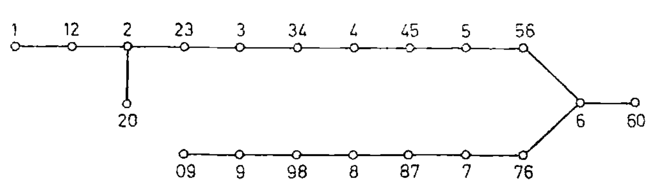
has the spectrum 6^1, 4^5, 1^4, and

$$G := \begin{bmatrix} 2I & -N \\ -N^t & 2I \end{bmatrix}$$

has spectrum $(2+\sqrt{6})^1$, 4^5, 3^4, 2^5, 1^4, 0^5, $(2-\sqrt{6})^1$. Therefore G is the Gram matrix of 25 vectors in $\mathbf{R}^{19,1}$. The corresponding reflexive graph is the incidence graph of the Petersen graph. The 25 vectors in $\mathbf{R}^{19,1}$ can be given integral coordinates as follows. Label the Petersen graph by use of a Hamiltonian path:



Let $[k\perp l\perp m]$ be the tree consisting of a path of length $k+l+m+2$ and two further nodes joined with the $(k+1)$st and $(k+l+2)$nd node of the path, respectively. The incidence graph of the Petersen graph contains the following subtree $[2\perp 7\perp 7]$ on 20 vertices.



The vertices of this subtree correspond to the vectors of a basis for $\mathbf{R}^{19,1}$. Indeed, elementary calculations show that the Cartan matrix of any $[2\perp 7\perp m]$ has determinant -4, since

$$\det \text{Cartan}\,[n] = n+1, \quad \det \text{Cartan}\,[2\perp n] = 5-n.$$

In terms of the orthonormal basis $\mathbf{e}_0$; $\mathbf{e}_1, \ldots, \mathbf{e}_{19}$ we may represent the vertices of the path by $\mathbf{e}_2-\mathbf{e}_1$, $\mathbf{e}_3-\mathbf{e}_2$, $\mathbf{e}_4-\mathbf{e}_3$, $\ldots$, $\mathbf{e}_{18}-\mathbf{e}_{17}$, $\mathbf{e}_{19}-\mathbf{e}_{18}$, vertex 20 by $\mathbf{e}_0+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$, and 60 by $3\mathbf{e}_0+\mathbf{e}_1+\ldots+\mathbf{e}_{11}$. Then the coordinates of the five remaining vectors are as follows:

$$18 = 4\mathbf{e}_0+2\mathbf{e}_1+\mathbf{e}_2+\ldots+\mathbf{e}_{15},$$

$$49 = 6\mathbf{e}_0+2(\mathbf{e}_1+\ldots+\mathbf{e}_7)+\mathbf{e}_8+\ldots+\mathbf{e}_{17},$$

$$0 = 4\mathbf{e}_0+\mathbf{e}_1+\ldots+\mathbf{e}_{18},$$

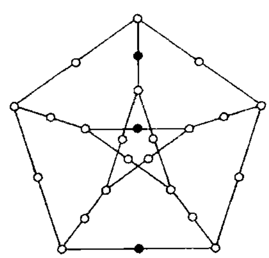
$$15 = 7\mathbf{e}_0+3\mathbf{e}_1+2(\mathbf{e}_2+\ldots+\mathbf{e}_9)+(\mathbf{e}_{10}+\ldots+\mathbf{e}_{19}),$$

$$37 = 9\mathbf{e}_0+3(\mathbf{e}_1+\ldots+\mathbf{e}_5)+2(\mathbf{e}_6+\ldots+\mathbf{e}_{13})+(\mathbf{e}_{14}+\ldots+\mathbf{e}_{19}).$$

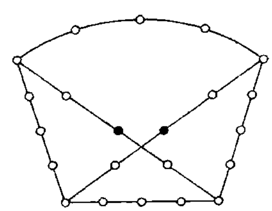
From the 25 vectors in $\mathbf{R}^{19,1}$, by deletion of the vectors involving $\mathbf{e}_{19}, \mathbf{e}_{18}, \ldots, \mathbf{e}_{d+1}$, we obtain a set of vectors in $\mathbf{R}^{d,1}$ of the size $n(d)$ given in the following table.

d	19	18	17	16	15	14	13	12
$n(d)$	25	22	20	18	17	15	14	13

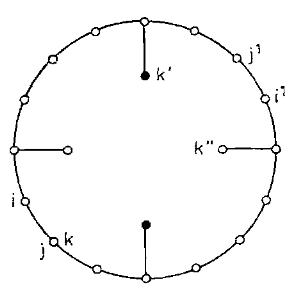
The corresponding graphs are the following (each next graph is obtained by deleting the black vertices):



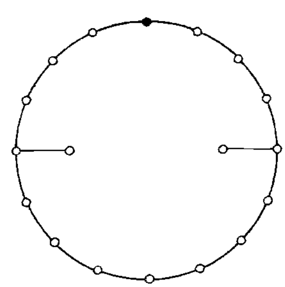
25 in $\mathbb{R}^{19,1}$



22 in $\mathbb{R}^{18,1}$



20 in $\mathbb{R}^{17,1}$

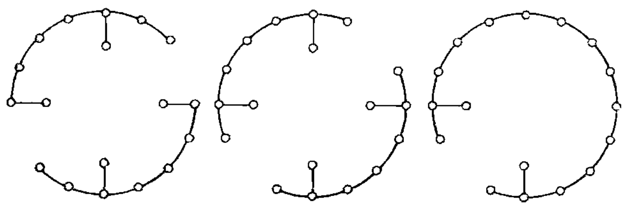


18 in $\mathbb{R}^{16,1}$

17 in $\mathbf{R}^{15,1}$: $[3\perp 7\perp 3]$; 15 in $\mathbf{R}^{14,1}$: $[2\perp 7\perp 2]$;
14 in $\mathbf{R}^{13,1}$: $[2\perp 7\perp 1]$; 13 in $\mathbf{R}^{12,1}$: $[2\perp 9]$.

In $\mathbf{R}^{19,1}$ the set of the 25 vectors cannot be extended by a vector of norm 2 having inner products 0 and -1. In $\mathbf{R}^{18,1}$ the 22 vectors, and in $\mathbf{R}^{17,1}$ the 20 vectors cannot be extended. This was proved in [25, 40]. Below we will sketch a matrix proof for these properties.

Vinberg also observed [38] that the 20 vectors in $\mathbf{R}^{17,1}$ give rise to various extended Dynkin diagrams in Euclidean $\mathbf{R}^{16}$ as follows:



delete i, i'	delete j, j'	delete k, k', k''
18 in $\mathbf{R}^{16}$	18 in $\mathbf{R}^{16}$	17 in $\mathbf{R}^{16}$
$\tilde{E}_8+\tilde{E}_8$	$\tilde{D}_8+\tilde{D}_8$	$\tilde{D}_{16}$

A general class of reflexive graphs is provided by the $\frac{1}{6}v(v-1)$ triples from a Steiner triple system on v points, yielding $\frac{1}{6}v(v-1)$ vectors of type $(1;\, 1^3 0^{v-3})$, orthogonal to $(3;\, 1^v)$ at 90° and 120° in $\mathbf{R}^{v-1,1}$.

A further example is provided by the 77 blocks of the Steiner system $3-(22, 6, 1)$, yielding 77 vectors of type $(\sqrt{2};\, 1^6 0^{16})$ orthogonal to $(3\sqrt{2};\, 1^{22})$. Since the block intersections are 2 and 0, the Gram matrix of these vectors is $2(2I-A)$, where A is the adjacency matrix of the corresponding strongly regular graph. Further examples are provided by the complements of other strongly regular graphs with smallest eigenvalue -3, cf. [16, 17]. The desire to find all such graphs has been one of the motivations for the present work.

Reflexive trees have been characterized by Maxwell [24] in the nonsingular, and by Neumaier [27] in the general case. It turns out that a hyperbolic tree contains a vertex, or an edge, whose deletion leaves a union of Euclidean trees. In view of the examples mentioned at the end of Section 4, it would be interesting to know which reflexive trees can be extended to (strongly) regular graphs with $\lambda_2 \leqq 2$.

Graph-extension problems of the kind mentioned above amount to the following. Find a set of n vectors $\mathbf{z}_1, \ldots, \mathbf{z}_n$ in $\mathbf{R}^{p,1}$ with norm 2 and inner products 0 and -1, which contains a given subset of m vectors $\mathbf{z}_1, \ldots, \mathbf{z}_m$ with these properties. In other words, given a graph A on m vertices with $\lambda_2 \leqq 2$, find a graph $\begin{bmatrix} A & B \\ B^t & C \end{bmatrix}$ on n vertices with second eigenvalue $\leqq 2$ such that

$$\begin{bmatrix} 2I-A & -B \\ -B^t & 2I-C \end{bmatrix} = \begin{bmatrix} 2I-A & (2I-A)S \\ S^t(2I-A) & S^t(2I-A)S \end{bmatrix}$$

for some $m\times(n-m)$ matrix S. We consider two special cases.

First, let $2I-\mathrm{A}$ be nonsingular of size $m=p+1$ and let $\begin{bmatrix} A & B \\ B^t & C \end{bmatrix}$ be regular of size n and valency k. Put $d:=(2-k)/n$. Then necessarily we have

$$(*) \qquad j^t(2I-A)^{-1}j = d^{-1}.$$

Indeed, $\mathbf{w} := (\mathbf{z}_1 + \ldots + \mathbf{z}_n)/n$ satisfies $(\mathbf{w}, \mathbf{z}_i) = (\mathbf{w}, \mathbf{w}) = d$ for $i = 1, \ldots, n$. On the other hand, the inner products of $\mathbf{w}$ with $\mathbf{z}_1, \ldots, \mathbf{z}_{p+1}$ form the column $dj = (2I - A)s$, hence

$$(\mathbf{w}, \mathbf{w}) = s^t(2I - A)s = dj^t s.$$

Then $(*)$ follows from $1 = j^t s = dj^t(2I - A)^{-1}j$. Notice that $(*)$ implies that the complement $\bar{A} = J - I - A$ has all eigenvalues ≥ -3. Indeed,

$$(2I - A - dJ)s = dj - djj^t s = 0,$$

and since $2I - A$ has signature $(p, 1)$ it follows that $2I - A - dJ$ is positive semidefinite, and that $2I - A + J = 3I + (J - I - A)$ is positive definite, whence our claim. On the basis of results similar to the above, and by use of a computer, attempts are being made by Bussemaker and Neumaier to construct new strongly regular graphs.

Secondly, we indicate why the 25 vectors in $\mathbf{R}^{19,1}$ (the incidence graph of the Petersen graph) cannot be extended. Suppose there exist (0, 1) vectors v of size 10 and x of size 15 such that the similar matrices

$$\begin{bmatrix} 2I_{10} & -N & v \\ -N^t & 2I_{15} & x \\ v^t & x^t & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2I - \frac{1}{2}NN^t & 0 & v + \frac{1}{2}Nx \\ 0 & 2I_{15} & 0 \\ v^t + \frac{1}{2}x^t N^t & 0^t & 2 - \frac{1}{2}x^t x \end{bmatrix}$$

of size 26 have signature (19,1). Then, by use of $NN^t = 3I + P$, the matrix

$$\begin{bmatrix} I - P & 2v + Nx \\ 2v^t + x^t N^t & 4 - x^t x \end{bmatrix}$$

of size 11 must have signature (4, 1), where P denotes the adjacency matrix of the Petersen graph, of size 10. Writing N and P in block form by use of a permutation matrix of size 5, we readily arrive at a contradiction.

6. Unimodular Euclidean lattices

An integral lattice (cf. [26], [35]) of signature (p, q) and dimension $p+q$ is a free $\mathbf{Z}$-module L with $p+q$ (free) generators together with a nondegenerate real symmetric bilinear form $(.,.)$ of signature (p, q) such that any two vectors $x, y \in L$ have integral inner product (x, y); L is called even if the norm (x, x) is even for every $x \in L$. The determinant of the Gram matrix $((x_i, x_j))$ of a basis $x_1, \ldots, x_{p+q}$ of L is called the discriminant of L; if this is ± 1, L is called unimodular. We call a lattice of signature (p, q) Euclidean if $q = 0$, and Lorentzian if $q = 1$.

The objects of the previous sections are related to integral lattices in various ways. For a positive integer r, and for a symmetric integral matrix M of size n with zero diagonal, the matrix $rI + M$ considered as a Gram matrix defines n vectors in $\mathbf{R}^{p,q}$ whose integral linear combinations define an integral lattice. In particular, to a graph with adjacency matrix A, we associate the r^+-lattice, defined by $rI + A$, and the r^--lattice, defined by $rI - A$.

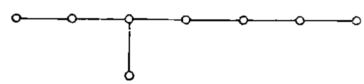
In the present section we consider three important unimodular integral Euclidean lattices: the Korkine—Zolotareff lattice E_8, the lattice $A_{19} := E_6^3$ [3] (cf. Conway and Sloane [8]), and the Leech lattice A_{24}. They will occur again in Section 7.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_8$ denote an orthonormal basis in $\mathbf{R}^8$. The integral lattices $\mathbf{Z}_8$, D_8, E_8 are defined by their generators as follows:

$$\mathbf{Z}_8 := \langle \mathbf{e}_1, \ldots, \mathbf{e}_8 \rangle_{\mathbf{Z}}, \quad D_8 := \langle \mathbf{e}_i + \mathbf{e}_j | i, j = 1, \ldots, 8 \rangle_{\mathbf{Z}}$$

$$E_8 := \langle D_8, \tfrac{1}{2}(\mathbf{e}_1 + \ldots + \mathbf{e}_8) \rangle_{\mathbf{Z}}.$$

E_8 is unimodular since D_8 is a sublattice of index 2 in both E_8 and $\mathbf{Z}_8$. Moreover E_8 is even, and its $112+128=240$ vectors of minimum norm 2 are $\pm\mathbf{e}_i \pm \mathbf{e}_j$ $(i \neq j = 1, \ldots, \ldots, 8)$, and $1/2(\pm\mathbf{e}_1 \pm \ldots \pm \mathbf{e}_8)$ with an even number of minus signs. E_8 is the 2^--lattice of the following Dynkin diagram:



In addition, E_8 is the 2^--lattice of a graph having least eigenvalue -2 of comultiplicity 8, which is not a generalized line graph (cf. [6]; the generalized line graphs generate only lattices of type D_n).

Any Euclidean lattice generated by norm 2 vectors is the 2^--lattice of a forest consisting of spherical trees of the types A_n, D_n, E_6, E_7, E_8. In fact, the irreducible Euclidean lattices generated by norm 2 vectors are the following:

$$A_n = \{\mathbf{x} \in \mathbf{Z}^{n+1} | x_1 + \ldots + x_{n+1} = 0\}, \quad D_n = \{\mathbf{x} \in \mathbf{Z}^n | x_1 + \ldots + x_n \text{ even}\},$$

$$E_8 (\text{coordinates as above}), \quad E_7 = \{\mathbf{x} \in E_8 | x_1 + \ldots + x_8 = 0\},$$

$$E_6 = \{\mathbf{x} \in E_8 | x_1 + \ldots + x_6 = x_7 + x_8 = 0\}.$$

This follows from Witt [41]; a proof in combinatorial terms follows from Cameron c.s. [6].

The lattices E_n have discriminant $9-n$, for $n=8, 7, 6$. Therefore, the lattice $E_6 \oplus E_6 \oplus E_6 \oplus \sqrt{3}\mathbf{Z}$ in $\mathbf{R}^{19}$ has discriminant 3^2. However, from this lattice we may obtain the unimodular integral lattice $\Lambda_{19} = E_6^3[3]$ by adding glue vectors as follows, cf. [8]. Select any $\mathbf{a} = \frac{1}{3}(1^4(-2)^2; 0^2)$ in the 6-dimensional subspace $x_7 = x_8 = 0$ of $\mathbf{R}^8$. The lattice Λ_{19}, generated over $\mathbf{Z}$ by $E_6 \oplus E_6 \oplus E_6 \oplus \sqrt{3}\mathbf{Z}$ and by $(\mathbf{0}; \mathbf{a}; -\mathbf{a}; 1/\sqrt{3})$, $(-\mathbf{a}; \mathbf{0}; \mathbf{a}; 1/\sqrt{3})$ is easily seen to be integral. Since $3\mathbf{a} \in E_6$ and $3 \cdot 1/\sqrt{3} = \sqrt{3}$, we have

$$[\Lambda_{19} : E_6 \oplus E_6 \oplus E_6 \oplus \sqrt{3}\,\mathbf{Z}] = 3^2.$$

Therefore, Λ_{19} is a unimodular integral lattice in $\mathbf{R}^{19}$. Interestingly, Λ_{19} is not spanned by vectors of norm 2 since the only such vectors are in $E_6 \oplus E_6 \oplus E_6 \subset \mathbf{R}^{18}$. However, Λ_{19} contains $6 \cdot 27^2 = 4374$ vectors of norm 3 spanning $\mathbf{R}^{19}$, namely those of type $\pm(\mathbf{0}; \mathbf{x}; -\mathbf{y}; 1/\sqrt{3})$ and cyclic permutations of the three E_6, where $\mathbf{x}$ and $\mathbf{y}$ are independently chosen from the $15 + 2 \times 6 = 27$ vectors of the types $\frac{1}{3}(1^4, (-2)^2; 0^2)$ and $\frac{1}{6}(1^5, (-5); \pm 3, \mp 3)$. It would be interesting to know whether this lattice is the 3^--lattice of a graph with largest eigenvalue 3 (perhaps a cubic graph on 20 vertices), or the 3^+-lattice of a graph with smallest eigenvalue -3, possibly a strongly regular graph with spectrum $k^1r^{18}(-3)^{n-19}$. Similar questions with 4 instead of 3 may be posed in connection with the Leech lattice which we are about to define.

In $\mathbf{R}^{24}$ let G_{24} be the lattice generated by $\sqrt{2}D_{24}$ and the vectors $\mathbf{x}/\sqrt{2}$ where $\mathbf{x}$, with coordinates 0 and 1, runs through the extended binary Golay code (cf. e.g. 16]). Since this code has dimension 12 and all its codewords have even weights we have $\mathbf{x}\in D_{24}$, $[G_{24}: \sqrt{2}D_{24}]=2^{12}$. Since D_{24} has discriminant 2, $\sqrt{2}D_{24}$ and G_{24} have discriminant $2^{12}\cdot 2$ and 2, respectively. Now take the integral linear combinations of the vectors of G_{24} and either

$$\mathbf{u}_L := (-3, 1^{23})/\sqrt{8} \quad \text{resp.} \quad \mathbf{u}_N := \sqrt{2}(1, 0^{23}).$$

In both cases $2\mathbf{u}\in G_{24}$, $(\mathbf{u}, \mathbf{u})\in 2\mathbf{Z}$, $(\mathbf{u}, \mathbf{x})\in \mathbf{Z}$ for $\mathbf{x}\in$Golay, hence the lattices are unimodular and even. In the first case, no vectors of norm 2 exist, and we have the Leech lattice Λ_{24}, cf. [21]. In the second case, the only norm 2 vectors are $\pm\sqrt{2}\mathbf{e}_i$, and we have the Niemeier lattice A_1^{24}, cf. [29]. It was shown in [29] that in $\mathbf{R}^{24}$ there exist precisely 24 integral unimodular even lattices, namely the Leech lattice Λ_{24}, whose vectors $\neq\mathbf{0}$ have minimum norm 4, and the 23 Niemeier lattices, whose vectors $\neq\mathbf{0}$ have minimum norm 2. From

$$2(\mathbf{x}, \mathbf{y}) = N(\mathbf{x})+N(\mathbf{y})-N(\mathbf{x}-\mathbf{y})$$

it follows that the angle φ between distinct lines spanned by minimal vectors satisfies $\cos\varphi\in\{0, \frac{1}{2}, \frac{1}{4}\}$ for the Leech lattice, and $\cos\varphi\in\{0, \frac{1}{2}\}$ for the Niemeier lattices. For the Leech lattice these lines are $\binom{28}{5}$ in number, the maximum possible for the given number of angles [14]. For the 23 Niemeier lattices these lines constitute unions of root systems, which characterize the various lattices. In fact, any Niemeier lattice consists of characterising lattice components of types A_n, D_n, E_n, refined into a unimodular lattice by use of certain glue vectors, cf. [8, 36]. The lattice A_1^{24} constructed above is one of them.

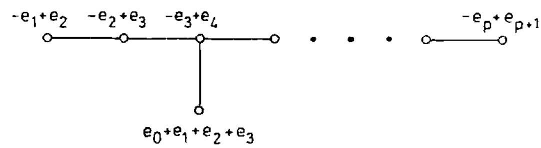
Minimum norm 4 means packing radius 1. Recently, Conway, Parker and Sloane [7] proved that the Leech lattice has covering radius $\sqrt{2}$, that is, congruent spheres around the lattice points which cover $\mathbf{R}^{24}$ have smallest radius $\sqrt{2}$. Let us take this fact as a starting point. A (deepest) hole is a vector $\mathbf{c}\notin \Lambda_{24}$ at distance $\sqrt{2}$ from a set $P=\{\mathbf{v}_1, \ldots, \mathbf{v}_v\}$ of nearest lattice points in Λ_{24}. Since $N(\mathbf{v}_i-\mathbf{c})=2$, the norm equation above implies

$$N(\mathbf{v}_i-\mathbf{v}_j)\in\{4, 6, 8\}, \quad (\mathbf{v}_i-\mathbf{c}, \mathbf{v}_j-\mathbf{c})\in\{0, -1, -2\},$$

for $i\neq j\in\{1, \ldots, v\}$. Therefore, the Gram matrix of the vectors $\mathbf{v}_i-\mathbf{c}$ is a Cartan matrix corresponding to a disjoint union of extended Dynkin diagrams of the types A_n, D_n, E_n. It turns out [7, 9] that Λ_{24} has 23 inequivalent holes, one for each Niemeier lattice, and that each of the 23 classes of holes (Niemeier lattices) gives rise to a construction of the Leech lattice.

7. Unimodular lattices of signature $(p, 1)$

In $\mathbf{R}^{p+1,1}$ with orthonormal basis $\mathbf{e}_0; \mathbf{e}_1, \ldots, \mathbf{e}_{p+1}$ the vector $\mathbf{w} := := 3\mathbf{e}_0 + \mathbf{e}_1 + \ldots + \mathbf{e}_{p+1}$ is orthogonal to each of the $p+1$ vectors indicated at the vertices of the following graph:



These vectors have the Gram matrix $2I-A$, of determinant $8-p$, where A is the adjacency matrix of the graph. The integral lattice generated by these vectors was considered by Du Val [15]. The lattice is even since the norm of all generating vectors is even. The lattice is unimodular in two cases. For $p=7$ it is the lattice E_8 in Euclidean $\mathbf{R}^8$. For $p=9$ it is a lattice of signature $(9, 1)$, to be denoted by $E_{9,1}$. Among the vectors of minimum norm 2 in $E_{9,1}$ we find 330 vectors of shape $\pm(1; 1^3 0^7)$ and $(0; 1(-1)0^8)$. These form 165 lines at 90° and 60° in $\mathbf{R}^{9,1}$, meeting the bound discussed in Section 2.

The lattices $E_{9,1}$ and E_8 are related by $E_{9,1} = E_8 \oplus \langle \mathbf{u}, \mathbf{v} \rangle$, where $\langle \mathbf{u}, \mathbf{v} \rangle$ is a hyperbolic plane with Gram matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Indeed, in the above representation for $E_{9,1}$ and E_8 the vectors $\mathbf{u} = (3; 1^8, 1, 0)$ and $\mathbf{v} = -(3; 1^8, 0, 1)$ apply. Likewise, for any integral [even] unimodular Euclidean lattice E in $\mathbf{R}^{p-1}$ there exists a lattice $H = E \oplus \langle \mathbf{u}, \mathbf{v} \rangle$ of signature $(p, 1)$ which is integral [even] unimodular. Indeed, take $\langle \mathbf{u}, \mathbf{v} \rangle \perp \mathbf{R}^{p-1}$ with Gram $(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ for the integral, and $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for the even case. Vinberg [38] has shown how conversely, on the basis of a given integral unimodular lattice H of signature $(p, 1)$, various integral unimodular lattices in $\mathbf{R}^{p-1}$ can be constructed. The method essentially amounts to stereographic projection in p-space with respect to the north pole $\langle \mathbf{u} \rangle$ of the sphere $x_1^2 + \ldots + x_p^2 = x_0^2$. It may be described as follows. Let $\mathbf{u}$ denote an indivisible isotropic vector of the given integral unimodular lattice H of signature $(p, 1)$. Let $\mathbf{u}^\perp$ denote the orthogonal complement of $\mathbf{u}$ in H. Then $E(\mathbf{u}) := \mathbf{u}^\perp / \langle \mathbf{u} \rangle$ is an integral unimodular lattice in Euclidean $\mathbf{R}^{p-1}$, of the same parity as H. Indeed, since $\mathbf{u}$ is indivisible and H is unimodular, there exists $\mathbf{v} \in H$ with $(\mathbf{u}, \mathbf{v}) = 1$ and $(\mathbf{v}, \mathbf{v}) \in \{0, 1\}$. By taking suitable linear combinations we obtain $H = \langle \mathbf{u}, \mathbf{v} \rangle \oplus E$, where E is unimodular in $\mathbf{R}^{p-1}$ and isomoprhic to $E(\mathbf{u})$. This proves our claim. By a similar argument, $E(\mathbf{u})$ and $E(\mathbf{u}')$ are isomorphic iff the isotropic vectors $\mathbf{u}$ and $\mathbf{u}'$ are in the same orbit of the automorphism group Aut H of the lattice H. Moreover, Aut $E(\mathbf{u})$ is isomorphic to the stabilizer of $\mathbf{u}$ in Aut H.

To exploit these ideas, we use the well-known fact [26, 35] that indefinite integral unimodular lattices are uniquely determined by their signature and their perity. Moreover, the even case only occurs for signature (p, q) with $p \equiv q \pmod 8$. A standard form for the odd case is provided by the lattice $I_{p,q} := \mathbf{Z}^{p,q}$. For the even case, following Conway and Sloane, we put (for $p \equiv q \pmod 8$ only)

$$\mathrm{II}_{p,q} := \langle D_{p,q}, \mathbf{a} \rangle_{\mathbf{Z}},$$

where $\mathbf{a}=\frac{1}{2}(1^p; 1^q)$ and $D_{p,q}=\{\mathbf{x}\in\mathbf{Z}^{p,q}|\sum x_i\in 2\mathbf{Z}\}$. Since $(\mathbf{a},\mathbf{a})=(p-q)/4$ and $(\mathbf{a},\mathbf{x})\in\mathbf{Z}$ for $\mathbf{x}\in D_{p,q}$ the lattice $\mathrm{II}_{p,q}$ is even and integral; it is unimodular since both $\mathrm{II}_{p,q}$ and $\mathbf{Z}^{p,q}$ have index 2 in $D_{p,q}$.

In particular, any integral unimodular lattice H of signature $(p, 1)$ is isomorphic to $I_{p,1}$ or (for $p\equiv 1 \mod 8$ only) to $\mathrm{II}_{p,1}$. The crucial observation by Vinberg [38] is that *all* unimodular Euclidean lattices of dimension $p-1$ embed in the above way into a *known* lattice of $\mathbf{R}^{p,1}$, namely into $\mathrm{I}_{p,1}$ (integral case) or into $\mathrm{II}_{p,1}$ (even case). Hence the classification of integral [even] Euclidean lattices turns out to be equivalent ot the classification of orbits of isotropic vectors in $\mathrm{I}_{p,1}$ [resp. $\mathrm{II}_{p,1}$].

Vinberg draws some interesting consequences from these observations. For $p=17$ he reconstructs [38] Kneser's classification of the integral unimodular Euclidean lattices in $\mathbf{R}^{16}$ from the 20 vectors in $\mathbf{R}^{17,1}$ of norm 2 with angles 90° and 120° which were mentioned in Section 5. Indeed, the lattices correspond to the extended spherical diagrams of rank 16, mentioned in Section 5 as sets of vectors in $\mathbf{R}^{16}$ at angles 90° and 120°. The same argument leads to the classification of 17- and 18-dimensional unimodular Euclidean lattices in [25, 40].

A second consequence deals with discrete groups generated by reflections. A *root* of a lattice L is a nonzero, nonisotropic vector $\mathbf{e}\in L$ such that the reflection $\mathbf{x}\to\mathbf{x}-2(\mathbf{e}, x)/(\mathbf{e},\mathbf{e})\mathbf{e}$ in the hyperplane orthogonal to $\mathbf{e}$ preserves L. For integral unimodular lattices L, the roots are just the vectors of norm ± 1 and ± 2 (or their multiples). A lattice L in $\mathbf{R}^{p,q}$ is called *reflexive* if its roots span $\mathbf{R}^{p,q}$, and if the group generated by root reflections has finite index in the group of all automorphisms of L (=orthogonal transformations fixing 0 and preserving L). For reflexive lattices the determination of the automorphism group and its fundamental domain becomes tractable. In this context the following results are obtained, cf. [38, 25, 40]:

(i) all unimodular integral Euclidean lattices of dimension $\leqq 18$ are reflexive;
(ii) unimodular integral lattices in $\mathbf{R}^{p,1}$ are reflexive iff $p\leqq 19$.

Indeed, the observations above imply that if a unimodular integral lattice in $\mathbf{R}^{p,1}$ is reflexive, then every unimodular integral Euclidean lattice in $\mathbf{R}^{p-1}$ is reflexive. Hence (i) follows from the existence of the inextensible set of 25 vectors of norm 2 spanning $\mathbf{R}^{19,1}$, and its subsets. Moreover, (ii) follows from the existence of non-reflexive unimodular integral lattices in $\mathbf{R}^{p-1}$ for all $p\geqq 20$, such as the odd lattices $\Lambda_{19}\oplus\mathbf{Z}^{p-20}$ and the even lattices $\Lambda_{24}\oplus E_8^m$.

To illustrate the above, we concentrate on the even lattices $\mathrm{II}_{p,1}$ for $p\equiv 1$ (mod 8).

For $p=9$ we have $\mathrm{II}_{9,1}\cong E_{9,1}$ since both lattices are even unimodular. $\mathbf{u}=(3; 1^9)$ is an indivisible isotropic vector of $\mathrm{II}_{9,1}$, and $\mathbf{u}^\perp/\langle\mathbf{u}\rangle\cong E_8$, cf. the beginning of this section.

For $p=17$, Kneser's classification shows that $\mathrm{II}_{17,1}$ has only one orbit of indivisible isotropic vectors, each leading to the Euclidean lattice $E_8\oplus E_8$. A simple choice is $\mathbf{u}=(3; 1^9 0^8)$; then the first E_8 is represented inside $\mathbf{Z}^{8,1}$ and the second E_8 inside $\frac{1}{2}\mathbf{Z}^8$.

For $p=25$ the lattice $\mathrm{II}_{25,1}$ must contain an isotropic vector corresponding to the Leech lattice. Indeed, Conway and Sloane [10] show that $\mathbf{u}=(70; 0, 1, 2, \ldots, 23, 24)$ works, thus interpreting the equality

$$0^2+1^2+2^2+\ldots+23^2+24^2 = 70^2.$$

They also exhibit the isotropic vectors corresponding to various Niemeier lattices; for instance $\mathbf{u}=(5; 1^{25})$ yields the Niemeier lattice of type A_{24}. Thus $\mathrm{II}_{25,1}$ explains naturally the relations between the Leech lattice and the Niemeier lattices mentioned in Section 6. It is reasonable to expect that this approach will also lead to a new proof for Niemeier's classification of even 24-dimensional unimodular Euclidean lattices [29, 36].

The paper [10] also exhibits an isotropic vector for the embedding of the Leech lattice $\mathrm{I}_{25,1}$.

In the Leech lattice Λ_{24} in $\mathbf{R}^{24}$, $(\mathbf{x},\mathbf{x})\geqq 4$ for all nonzero $\mathbf{x}\in\Lambda_{24}$, and $(\mathbf{y},\mathbf{y}')\leqq 2$ for all distinct norm 4 vectors $\mathbf{y},\mathbf{y}'\in\Lambda_{24}$. Moreover, Λ_{24} contains the 24 vectors $((-3)^1 1^{23})/\sqrt{8}$ having the Gram matrix $4I+2(J-I)$. Neumaier [28] calls a Euclidean lattice Λ in $\mathbf{R}^p$ of strict simplex type if $(\mathbf{x},\mathbf{x})\geqq n$ for all nonzero $\mathbf{x}\in\Lambda$ and $(\mathbf{y},\mathbf{y}')\leqq m$ for all distinct norm n vectors $\mathbf{y},\mathbf{y}'\in\Lambda$, and if Λ contains vectors $\mathbf{x}_1,\dots,\mathbf{x}_p$ having the Gram matrix $nI+m(J-I)$. The key observation is that we can change n into 1 and m into 0 by lifting $\mathbf{R}^p$ into Lorentz space $\mathbf{R}^{p,1}$. Indeed, adjoin $\mathbf{w}\perp\mathbf{R}^p$ with $(\mathbf{w},\mathbf{w})=-(n-m)/m$, thus obtaining the Lorentz space $\mathbf{R}^{p,1}$. The lattice

$$H:=\frac{1}{\sqrt{n-m}}\Lambda\oplus\frac{m}{n-m}\mathbf{Z}\mathbf{w}$$

contains the vectors $\mathbf{z}_1,\dots,\mathbf{z}_p$ defined by

$$\mathbf{z}_i=\frac{1}{\sqrt{n-m}}\mathbf{x}_i-\frac{n}{n-m}\mathbf{w}.$$

It is easy to check that indeed $\mathbf{z}_1,\dots,\mathbf{z}_p$ are orthonormal, that they are contained in the set $H_{1,1}$, where

$$H_{\alpha,\beta}:=\{\mathbf{z}\in H\,|\,(\mathbf{z},\mathbf{z})=\alpha,\ (\mathbf{z},\mathbf{w})=\beta\},$$

and that $(\mathbf{z},\mathbf{z}')\leqq 0$ for all $\mathbf{z},\mathbf{z}'\in H_{1,1}$, $\mathbf{z}\neq\mathbf{z}'$.

The sets $H_{\alpha,\beta}$ are finite. For instance,

$$\mathbf{x}=\mathbf{z}\sqrt{n-m}+\frac{m}{\sqrt{n-m}}\mathbf{w}$$

establishes a one-to-one correspondence between $H_{1,1}$ and the finite set of all norm n vectors of the Euclidean lattice Λ. The sets $H_{\alpha,\beta}$ have first been used by Du Val [15] in the case $H=\mathbf{Z}^{p,1}$ to study reflection groups. He discusses the lattices E_8 $(p=8)$ and $E_{9,1}$ $(p=10)$ mentioned above. For $p\leqq 8$, the sets $H_{1,1}$ are just the sets $H^{(p)}$ of Section 4. Several other lattices of simplex type give rise to similar sequences of sets $H_{1,1}$, cf. Neumaier [28].

Added in proof. Based on [7] and [10], Conway and Sloane recently extended Vinberg's work to unimodular lattices of dimension $(p, 1)$, $p\leqq 25$. For this very interesting work see J. H. Conway, *J. Algebra* **80** (1983), 156—163 and J. H. Conway and N. J. A. Sloane, *Proc. R. Soc. Lond. A* **384** (1982), 233—258.

References

[1] E. M. Andreev, Intersection of plane boundaries of a polytope with acute angles, *USSR Math. Notes* **8** (1970), 761—764 (English transl.).

[2] E. Bannai, A. Blokhuis, Ph. Delsarte and J. J. Seidel, An addition theorem for hyperbolic space, *J. Combin. Theory*, to appear.

[3] E. Bannai and N. J. A. Sloane, Uniqueness of certain spherical codes, *Canad. J. Math.* **33** (1981) 437—449.

[4] A. Blokhuis, An upper bound for the cardinality of a set of equiangular lines in $\mathbf{R}^{d,1}$, *memo* **1981—08**, *Techn. Univ. Eindhoven.*

[5] N. Bourbaki, *Groupes et algèbres de Lie, Chap.* **4**, **5**, **6**, Hermann (1968).

[6] P. J. Cameron, J. M. Goethals, J. J. Seidel and E. E. Shult, Line graphs, root systems and elliptic geometry, *J. Algebra* **43** (1976), 305—327.

[7] J. H. Conway, R. A. Parker and N. J. A. Sloane, The covering radius of the Leech lattice, *Proc. R. Soc. Lond. A* **380** (1982), 261—290.

[8] J. H. Conway and N. J. A. Sloane, On the enumeration of lattices of determinant one, *J. Number Theory* **15** (1982), 83—94; *Europ. J. Combin.* **3** (1982), 219—231.

[9] J. H. Conway and N. J. A. Sloane, Twenty-three constructions for the Leeach lattice, *Proc. R. Soc. Lond. A* **381** (1982), 275—283.

[10] J. H. Conway and N. J. A. Sloane, Lorentzian forms for the Leech lattice, *Bull. Amer. Math. Soc.* **6** (1982), 215—217.

[11] H. S. M. Coxeter, Discrete groups generated by reflections, *Annals Math.* **35** (1934), 588—621.

[12] H. S. M. Coxeter, Extreme forms, *Canad. J. Math.* **3** (1951), 391—441.

[13] H. S. M. Coxeter, *Regular polytopes* (third edition), Dover (1973).

[14] P. Delsarte, J. M. Goethals and J. J. Seidel, Bounds for systems of lines, and Jacobi polynomials, *Philips Research Reports* **30** (1975) 91*—105*.

[15] P. Du Val, The Kantor group of a set of points, *Proc. London Math. Soc.* **42** (1937), 18—51.

[16] J. M. Goethals and J. J. Seidel, Strongly regular graphs derived from combinatorial designs, *Canad. J. Math.* **22** (1970), 597—614.

[17] J. M. Goethals and J. J. Seidel, The regular two-graph on 276 vertices, *Discrete Math.* **12** (1975), 143—158.

[18] M. Kneser, Erzeugung ganzzahliger orthogonaler Gruppen durch Spiegelungen, *Math. Ann.* **225** (1981), 453—462.

[19] J. L. Koszul, *Lectures on hyperbolic Coxeter groups*, Univ. Notre Dame (1967).

[20] E. Lander, *Symmetric designs and codes, London Math. Soc., Lecture Notes,* to appear.

[21] J. Leech, Notes on sphere packings, *Canad. J. Math.* **19** (1967), 251—267.

[22a] J. H. van Lint, Notes on Egoritsjev's proof of the van der Waerden conjecture, *Lin. Alg. Appl.* **39** (1981), 1—8.

[22b] D. I. Falikman, A proof of the van der Waerden conjecture on the permanent of a doubly stochastic matrix, *Math. Zametki* **29** (1981), 931—938 (Russian).

[23] J. H. van Lint and J. J. Seidel, Equilateral point sets in elliptic geometry, *Proc. KNAW, A* **69** (=*Indag. Math.* **28**) (1966), 335—348.

[24] G. Maxwell, Hyperbolic trees, *J. Algebra* **54**, (1978), 46—49.

[25] J. Meyer, Präsentation der Einheitengruppe der quadratischen Form $F(X)=-X_0^2+X_1^2+\dots+X_{18}^2$, *Arch. Math.* **29** (1977), 261—266.

[26] J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Springer (1973).

[27] A. Neumaier, The second largest eigenvalue of a tree, *Lin. Alg. Appl.* **46** (1982), 9—25.

[28] A. Neumaier, Lattices of simplex type, *Siam J. Algebr. Discr. Meth.* **4** (1983).

[29] H. V. Niemeier, Definite quadratische Formen der Dimension 24 und Diskriminante 1, *J. Number Theory* **5** (1973), 142—178.

[30] V. V. Nikulin, On factor groups of the automorphism groups of hyperbolic forms modulo subgroups generated by 2-reflections, *Soviet Math. Dokl.* **20** (1979), 1156—1158 (Engl. trl.).

[31] V. V. Nikulin, On arithmetic groups generated by reflections in Lobachevsky spaces, *Math. USSR Izvestija* **16** (1981), 573—601 (Engl. trl.).

[32] J. J. Seidel, Angles and distances in n-dimensional Euclidean and non-Euclidean geometry I, II, III, *Proc. KNAW, A* **58** (=*Indag. Math.* **17**) (1955), 329—335, 336—340, 535—541.

[33] J. J. Seidel, A survey of two-graphs, *Proc. Intern. Coll. Teorie Combinatorie* (Roma 1973), *Accad. Naz. Lincei* (1976), 481—511.

[34] J. J. Seidel, Strongly regular graphs, in: *Surveys in Combinatorics, Lecture Notes London Math. Soc.* (ed. Bollobás) **38** (1979), 157—180.

[35] J. P. SERRE, *Cours d'arithmétique*, Presses Univ, France (1970).
[36] B. B. VENKOV, The classification of integral quadratic forms, *Proc. Steklov Inst. Math.* **4** (1980), 63—74 (Engl. trl.).
[37] E. B. VINBERG, On groups of unit elements of certain quadratic forms, *Math. USSR Sbornik* **16** (1972), 17—35 (Engl. trl.).
[38] E. B. VINBERG, On unimodular integral quadratic forms, *Funct. Anal. Appl.* **6** (1972), 105—111 (Engl. trl.).
[39] E. B. VINBERG, Some arithmetical discrete groups in Lobacevskii spaces, in: *Proc. Intenr. Coll. Discrete subgroups of Lie groups and appl. to Moduli*, Bombay 1973, Oxford Univ. Press (1975), 323—347.
[40] E. B. VINBERG and I. M. Kaplanskaya, On the Groups $O_{18,1}(\mathbf{Z})$ and $O_{19,1}(\mathbf{Z})$, *Soviet Math. Dokl.* **19** (1978), 194—197, (Engl. trl.).
[41] E. WITT, Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe, *Abh. Math. Sem. Hamburg* **14** (1941), 289—322.

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